$$S = H\bar{S}H^T \tag{18}$$

From Eqs. (6) and (7) we see that

$$H = (V_2^I V_1^{-1}) \tag{19}$$

where I is an identity matrix of mth order. Inserting this in Eq. (18) and recalling Eq. (4) gives, finally, the simple formula

$$S = VG^{-1}V^T \tag{20}$$

for the unsupported stiffness matrix of the structural element.

The matrix G can be conveniently evaluated by numerical integration using a formula of the type

$$G = \sum_i \gamma_i U_i^T N U_i \tag{21}$$

where the γ_i are constants and U_i = the value of U at some point i.

References

[1] Argyris, J. H., *Energy Theorems and Structural Analysis, Part 1, General Theory*, Aircraft Engineering, Eq. (84), p. 43, Feb. 1955.

[2] Timoshenko, S., and Goodier, J. N., *Theory of Elasticity*, McGraw-Hill Book Co., Inc., 2nd Ed., p. 147, Eq. (c), 1951.

[3] Turner, M. J., Clough, R. W., Martin, H. C., and Topp, L. J., *Stiffness and Deflection Analysis of Complex Structures*, Journal of the Aeronautical Sciences, Vol. 23, No. 9, Sept., 1956.

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On Minor-Circle Turns

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With the advent of the space age, many people have become concerned with the maneuvering characteristics of space vehicles. A great deal of literature is available on the various aspects of orbital transfers and on the various methods of providing improved performance. In a recent paper W. H. T. Loh[1] introduced a definition of a minor-circle turn. We would like to suggest at this time an alternative definition which appears useful in that it removes some of the restrictions found in the work by Loh.

Loh's definition of a minor-circle turn requires the vehicle to fly in a plane which is oriented normal to the axis of the earth and which is elevated above the equator at that distance where the plane intersects the surface of the earth at the desired latitude. While this is satisfactory for near-earth trajectories and for gentle minor-circle turns, it does not appear to be capable of handling the cases where the altitudes are large and where the latitudes of the turn approach ninety degrees. To surmount this difficulty, let us define a minor-circle trajectory as that path where the vector pointing toward the center of the earth from the vehicle always cuts the surface of the earth at a constant latitude. This is equivalent to requiring the path of the vehicle to lie on the surface of a cone whose solid angle is the supplement of twice the latitude (see Fig. 1). With this definition, we are no longer restricted to near-earth trajectories or to gentle turns but may include sharp turns with steep re-entry angles without violating our assumptions. The equations of motion may be written by inspection if we choose our three directions as normal to the surface of the cone, tangent to the velocity vector lying in the cone, and perpendicular to these two directions. Defining γ as our ascent angle, α as our bank angle measured out of the cone, and λ as our latitude, we obtain

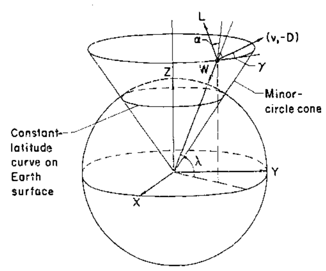

Fig. 1. Minor-circle cone.

$$(1/2)m(dv^2/ds) = m(dv/dt) = -D - mg \sin \gamma$$

$$mv^2(d\gamma/ds) = L \cos \alpha - mg \cos \gamma + (mv^2/r) \cos \gamma$$

$$(mv^2/r) \cos^2 \gamma \tan \lambda = L \sin \alpha$$

where s is the distance along the path.

These three equations may be compared to those of Loh and found to be similar but not identical. For near-earth orbits of slowly varying altitudes, the equivalence is clear as is expected from the geometrical considerations of the two definitions. It is also clear that this new definition adds no new restrictions and thus the results obtained by Loh in his paper may be reproduced if desired. To demonstrate this, his Eq. (2) becomes, with our definition and notation,

$$\frac{dv^2}{ds} + \frac{2g}{(L/D)} \sqrt{\cos^2 \gamma \left\{ \left[\frac{v^2}{gr} \cos \gamma \tan \lambda\right]^2 + \left[1 - \frac{v^2}{gr} + \frac{v^2}{g \cos \gamma}\left(\frac{d\gamma}{ds}\right)\right]^2 \right\}} = -2g \sin \gamma$$

and the aerodynamic control required at any moment along the minor circle becomes

$$\sin \alpha = \frac{(v^2/gr) \cos \gamma \tan \lambda}{\sqrt{\left[\frac{v^2}{gr} \cos \gamma \tan \lambda\right]^2 + \left[1 - \frac{v^2}{gr} + \frac{v^2}{g \cos \gamma}\left(\frac{d\gamma}{ds}\right)\right]^2}}$$

which is similar to Eq. (3) in Loh's paper.

Reference

[1] Loh, W. H. T., *Dynamics and Thermodynamics of Re-Entry*, Journal of Aerospace Sciences, Vol. 27, No. 10, Oct. 1960.

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Design of Tufts for Flow Visualization

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Tufts are frequently used for visualizing gas or liquid motions near solid surfaces. They are especially useful in develop-

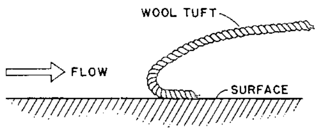

FIG. 1.

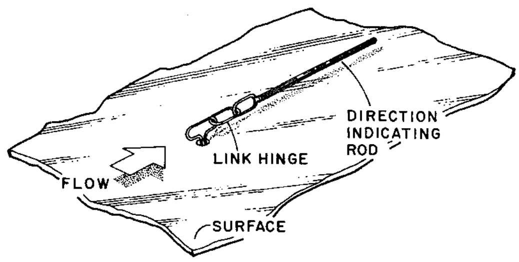

FIG. 2.

mental research for discovering regions of backflow, boundary-layer separation, and strong cross-flow, with the aim of correcting these conditions.

Commonly the tuft is a strand of wool, fixed at one end to the solid surface. Mechanically it is a cantilever beam with one end free and the other end fixed, and it assumes a position as in Fig. 1, exhibiting the flow direction at some distance from the surface. For the tuft to lie very close to the surface, as one would normally prefer, the bending stiffness must be exceedingly small. But this usually requires a fiber so fine that it is hard to see. Thus the requirements of flexibility and of ease of viewing are in conflict.

The ideal tuft would consist of two parts: a perfectly flexible universal hinge, and a rod-like direction indicator large enough to be easily visible and of a material neutrally buoyant in the fluid. Fig. 2 shows an approximation to this. The hinge is basically two or three links of a chain, made of very fine wire or nylon thread. The rod, for air, could be a wool tuft of suitable thickness; for water, it could be a dowel of wood or plastic having a specific gravity close to that of water.

In practice, flow-indicating tufts of this design have been very successful. The tufts move about freely and lie very close to the surface.

Thermal Stresses in an Elastic Half-Space With a Moving Boundary

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THE THERMAL STRESSES in an elastic semi-infinite solid whose boundary moves as a result of melting or freezing are considered in this note. The analysis is based on the uncoupled theory of thermal stresses. Such effects as viscosity and the variation of material properties with the temperature are also neglected. There is, of course, a question as to whether the elastic equations apply in phase transition conditions; the objective here is the investigation of the effect of inertia and comparing the result with quasi-static theory. Generally speaking, the neglect of inertia for problems *not involving a moving boundary* has been shown to be justified in Ref. 1, where a complete account of the various investigations is given. It will be shown that inertia plays an important role in moving boundary problems, and that its effect should be considered in certain cases.

Let us consider a solid occupying the region $x \geq 0$, initially unstressed and at a uniform temperature which shall be referred to as zero. The solid is to be melted by the application of heat at the plane $x = 0$. We assume that the solution of the temperature $T(x,t)$ and the free boundary position $s(t)$ is known. The thermal stresses σ_x, σ_y, and σ_z in directions x, y, and z are given in terms of the x-component of strain $e(x, t)$ by the equations

$$\left.\begin{aligned} \sigma_x &= \rho(c^2 e - \kappa T) \\ \sigma_y = \sigma_z &= (1-\nu)^{-1}(\nu\sigma_x - E\alpha T) \end{aligned}\right\} \tag{1}$$

where E, ν, ρ, and α are Young's modulus, Poisson's ratio, density, and coefficient of thermal expansion, respectively, $\kappa = E\alpha(1 - 2\nu)^{-1}\rho^{-1}$, and c is the velocity of dilational waves. The strain $e(x, t)$ is the solution to the following boundary-value problem

$$c^2 e_{xx} - e_{tt} = \kappa T_{xx} \quad \text{in } R\text{:} \quad s(t) < x < \infty,\ t > 0 \tag{2}$$

$$e(s(t), t) = \kappa c^{-2} T_m \tag{3}$$

$$e(\infty, t) = 0 \tag{4}$$

$$e(x, 0) = e_t(x, 0) = 0 \tag{5}$$

Here T_m is the melting temperature and condition (3) states that at the boundary $x = s(t)$ the stress should be zero.

To obtain the solution, we examine the state of affairs in the x-t plane. In those cases when $s(t) \sim t^{1/2}$ the region of interest R of the x-t plane is bounded by the curve $x = s(t)$ and the line $t = 0$. This region is divided into two subregions R_1 and R_2. The region R_1 is the locus of all points whose domain of dependence fall entirely within R_1 and do not intersect the moving boundary. Region R_2 is simply $R - R_1$. The two regions are separated by a semi-infinite characteristic which is tangent to the curve $x = s(t)$ at the point $(\bar{x}, \bar{t})$ and extends from this point in the direction of increasing x and t.

Thus, in region R_1 the solution e is equal to the function $e_0(x, t)$ which satisfies Eq. (2) and initial conditions (5) and is regular as $|x| \to \infty$. This solution may be obtained either by transform technique or from the formula

$$e_0 = \frac{\kappa}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} T_{xx}(\xi, \tau)\, d\xi\, d\tau \tag{6}$$

In region R_2 the initial conditions no longer matter, and the solution may be chosen as

$$e = e_0(x, t) + f(ct - x) \tag{7}$$

Application of the boundary condition (3) results in the functional relationship

$$f(\xi) = \kappa c^{-2} T_m - e_0(s(t(\xi)), t(\xi)) \qquad \text{where } \xi = ct - s(t) \tag{8}$$

This equation determines f for any argument ξ once t has been solved in terms of ξ from the second equation in (8). Thus, the solution is completely determined in terms of $e_0(x, t)$ and the stresses may be found from Eq. (1). We note that the stress is discontinuous across the characteristic that separates the two regions by the constant amount $\Delta\sigma_x = \rho c^2 f(-x_0)$, where x_0 is the intersection of this characteristic with the x axis.

We also note that for times corresponding to the portion $0 < t < \bar{t}$ of the boundary the stress-free condition cannot be met and the initial conditions predetermine the boundary stress. This is a consequence of the initially infinite speed of the boundary, and the conjecture is made that if the effect of thermoelastic coupling is included, the speed of the boundary will be less than the sound speed of the medium. This difficulty does not arise in freezing problems, in which case the boundary moves opposite to the direction of wave propagation.

We shall now apply the solution obtained above to the case of a semi-infinite solid which is being melted by keeping the plane $x = 0$ at constant temperature T_1, $(T_1 > T_m)$. This is the Neumann problem and its solution is[2]

$$T = T_m \operatorname{erfc}(x/2\sqrt{kt})/\operatorname{erfc} h,\ s = 2h(kt)^{1/2} \tag{9}$$